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A Linear Algebra Approach to Cyclic Extensions in Galois Theory

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ABSTRACT: We study cyclotomic field $Q(\xi_n)$, where ξ_n is the primitive n^{th} root of unity and it is shown that its Galois group isomorphic to $(Z/nZ)^*$ Norms, traces and related theorems such as Hilbert theorem 90 and trace theorem (Hilbert theorem 90 additive from) are proved . Theorems on cyclic extensions [[7], [8]] are proved by using Hilbert theorem 90 and trace theorem. Linear algebra approach is used to prove theorems on cyclic extensions.

KEYWORDS: Galois extension, trace, cyclic group, rank, nullity, irreducible polynomial.

I. INTRODUCTION

A beginning course in Galois theory often includes a discussion of cyclic extensions , that is, Galois extensions whose Galois groups are cyclic. The usual approach isto derive the results oncyclic extensionsas corollaries to the"norm "(Hilbert's "Theorem90" [1],[2]) and "trace" theorems [1] , [2] .In thispaper we proveHilbert's " Theorem 90 " by using linear algebra approach.

II. RELATED WORKS AND RESULTS

Theorem 1.1 :Let n be a positive integer, Let F be a field of characteristic zero or characteristicp with (p,n) = 1 and assume that F contains a primitive n^{th} root of unity ω . IfK/F is a Galois extension with cyclic Galois group of order n, then $K = F(\alpha)$, where α is a root of an irreducible polynomial $x^n - a \in F[x]$ and a be an element of F which is not a n^{th} power in F .

Proof :The Galois group G = G(K/F) is cyclic group of order n = [K:F] , $\therefore G = \langle \sigma \rangle$, $\sigma \neq$ identity .

We observe thatσsatisfies a polynomial$x^n - 1$.

Also σsatisfies nopolynomial of smallerdegree ,because if it satisfies ,then it contradict independence of the characters $1,\sigma,\sigma^2,.....,\sigma^{n-1}$; hence$x^n - 1$ is the minimal polynomial ofσ, indeed $x^n - 1$ is the characteristic polynomial of σ .Since ω^{-1} isthe root of $x^n - 1$, it is an eigenvalue of σ.(Notethat $\omega^{-1} \in F$)

If α is an eigenvector of σcorresponding to ω^{-1} then$_\sigma(\alpha) = \omega^{-1}\alpha$.

$\therefore \omega = \alpha\sigma(\alpha)^{-1}$ ....(1.1)

Where$\omega$is a primitive n^{th}root of unity , then$N(\omega) = \omega^n = 1$, because $\omega \in F$.

Now form (1.1),we have

$\omega = \alpha\sigma(\alpha)^{-1}$,for some$\alpha \in K$.

Hence $\sigma(\alpha) = \alpha\omega^{-1}$

$\therefore \sigma(\alpha^n) = (\alpha\omega^{-1})^n = \alpha^n(\omega^{-1})^n$, But $(\omega^{-1})^n = 1$

$\therefore \sigma(\alpha^n) = \alpha^n$

Since σ generates G , the element $\alpha^n$is in the fixed $K^G = F$ (becauseσgenerates G and K/ F is Galois extension).

Notethat$\alpha \in F$, because if $\alpha \in F$ then $\sigma(\alpha) = \alpha$.

$\therefore \omega = 1$, which is contradiction and there is at least one eigenvalue different from, thereforeour assumption is wrong.$\therefore \alpha \in F$

$\therefore F(\alpha) \neq F$ is an intermediate field, therefore K =F(α), because[K:F] = n , and hence Khas no proper intermediate field.

Hencetheproof.

Theorem 1.2 :LetF be a field of characteristics p and let K/F be Galois extension with cyclic Galois group of orderp. Then K = F(α), where $\alpha$is a root of an irreducible polynomial $x^p - x - a \in F[x]$.

Proof :The Galois group G = G (K/F) is a cyclic group of order p = [K:F].$\therefore G = \langle \sigma \rangle$, $\sigma \neq$identity .

Let us view K as a vector space over F.

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$\therefore \sigma : K \to K$ is a linear operator .
Let $\tau : K \to K$ is another linear operator, defined by
$\tau = \sigma^{-}$ identity
We wish to show that $1 \in Im(\tau)$.
Since K/F is a Galois extension.
$\therefore$ F isa fixed field of G .
$\therefore$ Ker τ =F (Since $F = \{x \in K \mid \sigma(x) = x$, for all $\sigma \in G\}$)
Claim :Im $(\tau) \cap \ker(\tau) \neq \{0\}$
Proof by contradiction.
Suppose ,Im $(\tau) \cap \ker(\tau) = \{0\}$
$\therefore K = Im(\tau) + \ker(\tau)$ (By Dimension Theorem).....(1.2)
Now,σ^p = identity
$\therefore \tau^p = \sigma^p$ – identity = 0
$\therefore \tau^p$ is a zero map.
Let $\beta \in K$,Byeq^n (1.2)
$\beta = \gamma + \tau(\delta)$, where $\gamma \in \ker(\tau)$.
Then ,$\tau^{p-1}(\beta) = \tau^{p-1}(\gamma) + \tau^p(\delta)$
Now,$\tau^p(\delta) = 0$, $\gamma \in \ker(\tau) . \therefore \tau^{P-1}(\gamma) = 0$
⇨ τ^{p-1} is a zero map.
Continuing inthis way, we get τ is a zero map.
Which iscontradiction.
$\therefore$ Our assumption is wrong.
$\therefore$ Im $(\tau) \cap \ker(\tau) \neq \{0\}$.
Since ker (τ) = F has dimension 1.
$\therefore$ Im $(\tau) \cap \ker(\tau) = F$.
$\therefore$ F$\subset$Im (τ)
In particular , $1 \in$Im (τ)
$\therefore 1 = \sigma(\alpha) - \alpha$
$\therefore \sigma(\alpha) = \alpha + 1$
Hence$\sigma^i(\alpha) = \alpha + i$, fori = 1 ,2 ,.....p
Since ch F = p ;α, $\alpha + 1$,....., $\alpha + p - 1$ are distinct.
Hence [F(α):F] = p; $\therefore$ K = F(α).
Consider, $\alpha^p - \alpha$
$\therefore \sigma(\alpha^p - \alpha) = (\sigma(\alpha))^p - \sigma(\alpha)$
$= (\alpha+1)^p - (\alpha+1)$
$= \alpha^p - \alpha$
$\therefore \alpha^p - \alpha \in K^{<\sigma>} = K^G = F$ (Fixed field of G)
Let a =$\alpha^p - \alpha \in F$
Then αsatisfies $f(x) = x^p - x - a = 0$.
The roots of f (x) are α, α+1,.....α+p –1 which are distinct.
$\therefore$ f (x) isirreducible polynomial $\in F[x]$.
Hence the proof .

Inthefollowing,trace theoremgiven in [[2] ,theorem 1.1]is proved by using just independence of characters and rank - nullity theorem.

Theorem 1.3 : Let K/F be a Galois extension with cyclic Galois group, generated by σ. Then for$\alpha \in$ K. Tr (α) = T $(\alpha) = 0$ if and only if $\alpha = \sigma(\beta) - \beta$for some$\beta \in$ K,Let us view K as a vector space over F and G = $< \sigma >$ is cyclic group generated by σ.
Define, $\tau = \sigma$ – identityandT : K $\to$K be the trace ,then both T and τ are linear operators on the K-vector space F , because trace is an additive transformation on K.
The proof of theorem is reduced, to showing that ,Ker (T) = Im (τ)
First we see that Im $(\tau) \subset$ker (T)
For ,Let$\alpha \in$Im (τ)

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$\therefore \alpha = \sigma(\beta) - \beta$ for some $\beta \in K$

⇨ $\alpha \in$ ker (T) (since ker (T) = $\{\alpha \in K | \alpha = \sigma(\beta) - \beta$ for $\beta \in K\}$)

$\therefore$ Im (τ) $\subset$ ker (T)……..…….(1.3)

Now,Since K / F is Galois extension,

We haveker (τ) = F has dimension 1.(as in theorem 1.2)

$\therefore$ Im (τ) is(n – 1) dimensional, where n = [K:F]

Now ,By independence of characters, Tis not zero map, so that ker (T) also has dimension (n – 1).(by1.3)

dim (ker T) = dim (Im (τ))

$\therefore$ ker (T) = Im (τ) .

Hence the proof.

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